



# Wiener Processes and Itô's Lemma

## Chapter 12

# Types of Stochastic Processes

- Discrete time; discrete variable
- Discrete time; continuous variable
- Continuous time; discrete variable
- Continuous time; continuous variable

# Modeling Stock Prices

- We can use any of the four types of stochastic processes to model stock prices
- The continuous time, continuous variable process proves to be the most useful for the purposes of valuing derivatives

# Markov Processes (See pages 259-60)

- In a Markov process future movements in a variable depend only on where we are, not the history of how we got where we are
- We assume that stock prices follow Markov processes

# Weak-Form Market Efficiency

- This asserts that it is impossible to produce consistently superior returns with a trading rule based on the past history of stock prices. In other words technical analysis does not work.
- A Markov process for stock prices is consistent with weak-form market efficiency

# Example of a Discrete Time Continuous Variable Model

- A stock price is currently at \$40
- At the end of 1 year it is considered that it will have a normal probability distribution of with mean \$40 and standard deviation \$10

# Questions

- What is the probability distribution of the stock price at the end of 2 years?
- $\frac{1}{2}$  years?
- $\frac{1}{4}$  years?
- $\Delta t$  years?

Taking limits we have defined a continuous variable, continuous time process

# Variances & Standard Deviations

- In Markov processes changes in successive periods of time are independent
- This means that variances are additive
- Standard deviations are not additive



# Variances & Standard Deviations (continued)

- In our example it is correct to say that the variance is 100 per year.
- It is strictly speaking not correct to say that the standard deviation is 10 per year.

# A Wiener Process (See pages 261-63)

- We consider a variable  $z$  whose value changes continuously
- Define  $\phi(\mu, \nu)$  as a normal distribution with mean  $\mu$  and variance  $\nu$
- The change in a small interval of time  $\Delta t$  is  $\Delta z$
- The variable follows a Wiener process if
  - $\Delta z = \varepsilon \sqrt{\Delta t}$  where  $\varepsilon$  is  $\phi(0, 1)$
  - The values of  $\Delta z$  for any 2 different (non-overlapping) periods of time are independent

□

# Properties of a Wiener Process

- Mean of  $[z(T) - z(0)]$  is 0
- Variance of  $[z(T) - z(0)]$  is  $T$
- Standard deviation of  $[z(T) - z(0)]$  is  $\sqrt{T}$

# Taking Limits . . .

- What does an expression involving  $dz$  and  $dt$  mean?
- It should be interpreted as meaning that the corresponding expression involving  $\Delta z$  and  $\Delta t$  is true in the limit as  $\Delta t$  tends to zero
- In this respect, stochastic calculus is analogous to ordinary calculus

# Generalized Wiener Processes

(See page 263-65)

- A Wiener process has a drift rate (i.e. average change per unit time) of 0 and a variance rate of 1
- In a generalized Wiener process the drift rate and the variance rate can be set equal to any chosen constants

# Generalized Wiener Processes

(continued)

The variable  $x$  follows a generalized Wiener process with a drift rate of  $a$  and a variance rate of  $b^2$  if

$$dx = a dt + b dz$$

# Generalized Wiener Processes

(continued)

$$\Delta x = a \Delta t + b \varepsilon \sqrt{\Delta t}$$

- Mean change in  $x$  in time  $T$  is  $aT$
- Variance of change in  $x$  in time  $T$  is  $b^2 T$
- Standard deviation of change in  $x$  in time  $T$  is  $b\sqrt{T}$

# The Example Revisited

- A stock price starts at 40 and has a probability distribution of  $\phi(40,100)$  at the end of the year
- If we assume the stochastic process is Markov with no drift then the process is

$$dS = 10dz$$

- If the stock price were expected to grow by \$8 on average during the year, so that the year-end distribution is  $\phi(48,100)$ , the process would be

$$dS = 8dt + 10dz$$



# Itô Process (See pages 265)

- In an Itô process the drift rate and the variance rate are functions of time

$$dx = a(x, t) dt + b(x, t) dz$$

- The discrete time equivalent  $\Delta x = a(x, t) \Delta t + b(x, t) \sqrt{\Delta t}$

is only true in the limit as  $\Delta t$  tends to zero

# Why a Generalized Wiener Process Is Not Appropriate for Stocks

- For a stock price we can conjecture that its expected percentage change in a short period of time remains constant, not its expected absolute change in a short period of time
- We can also conjecture that our uncertainty as to the size of future stock price movements is proportional to the level of the stock price

# An Ito Process for Stock Prices

(See pages 269-71)

$$dS = \mu S dt + \sigma S dz$$

where  $\mu$  is the expected return  $\sigma$  is the volatility.

The discrete time equivalent is

$$\Delta S = \mu S \Delta t + \sigma S \epsilon \sqrt{\Delta t}$$

# Monte Carlo Simulation

- We can sample random paths for the stock price by sampling values for  $\varepsilon$
- Suppose  $\mu = 0.15$ ,  $\sigma = 0.30$ , and  $\Delta t = 1$  week ( $= 1/52$  years), then

$$\Delta S = 0.00288 + 0.0416\varepsilon$$

# Monte Carlo Simulation – One Path

(See Table 12.1, page 268)

| Week | Stock Price at Start of Period | Random Sample for $\varepsilon$ | Change in Stock Price, $\Delta S$ |
|------|--------------------------------|---------------------------------|-----------------------------------|
| 0    | 100.00                         | 0.52                            | 2.45                              |
| 1    | 102.45                         | 1.44                            | 6.43                              |
| 2    | 108.88                         | -0.86                           | -3.58                             |
| 3    | 105.30                         | 1.46                            | 6.70                              |
| 4    | 112.00                         | -0.69                           | -2.89                             |

# Itô's Lemma (See pages 269-270)

- If we know the stochastic process followed by  $x$ , Itô's lemma tells us the stochastic process followed by some function  $G(x, t)$
- Since a derivative is a function of the price of the underlying and time, Itô's lemma plays an important part in the analysis of derivative securities

# Taylor Series Expansion

- A Taylor's series expansion of  $G(x, t)$  gives

$$\begin{aligned}\Delta G = & \frac{\partial G}{\partial x} \Delta x + \frac{\partial G}{\partial t} \Delta t + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} \Delta x^2 \\ & + \frac{\partial^2 G}{\partial x \partial t} \Delta x \Delta t + \frac{1}{2} \frac{\partial^2 G}{\partial t^2} \Delta t^2 + \dots\end{aligned}$$

# Ignoring Terms of Higher Order Than $\Delta t$

In ordinary calculus we have

$$\Delta G = \frac{\partial G}{\partial x} \Delta x + \frac{\partial G}{\partial t} \Delta t$$

In stochastic calculus this becomes

$$\Delta G = \frac{\partial G}{\partial x} \Delta x + \frac{\partial G}{\partial t} \Delta t + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} \Delta x^2$$

because  $\Delta x$  has a component which is of order  $\sqrt{\Delta t}$



# Substituting for $\Delta x$

Suppose

$$dx = a(x, t)dt + b(x, t)dz$$

so that

$$\Delta x = a \Delta t + b \varepsilon \sqrt{\Delta t}$$

Then ignoring terms of higher order than  $\Delta t$

$$\Delta G = \frac{\partial G}{\partial x} \Delta x + \frac{\partial G}{\partial t} \Delta t + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2 \varepsilon^2 \Delta t$$

# The $\varepsilon^2 \Delta t$ Term

Since  $\varepsilon \approx \phi(0,1)$ ,  $E(\varepsilon) = 0$

$$E(\varepsilon^2) - [E(\varepsilon)]^2 = 1$$

$$E(\varepsilon^2) = 1$$

It follows that  $E(\varepsilon^2 \Delta t) = \Delta t$

The variance of  $\Delta t$  is proportional to  $\Delta t^2$  and can be ignored. Hence

$$\Delta G = \frac{\partial G}{\partial X} \Delta X + \frac{\partial G}{\partial t} \Delta t + \frac{1}{2} \frac{\partial^2 G}{\partial X^2} b^2 \Delta t$$

# Taking Limits

Taking limits  $dG = \frac{\partial G}{\partial x} dx + \frac{\partial G}{\partial t} dt + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2 dt$

Substituting:  $dx = a dt + b dz$

We obtain  $dG = \left( \frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2 \right) dt + \frac{\partial G}{\partial x} b dz$

This is Ito's Lemma

# Application of Ito's Lemma to a Stock Price Process

The stock price process is

$$dS = \mu S dt + \sigma S dz$$

For a function  $G$  of  $S$  and  $t$

$$dG = \left( \frac{\partial G}{\partial S} \mu S + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial G}{\partial S} \sigma S dz$$

# Examples

1. The forward price of a stock for a contract maturing at time  $T$

$$G = S e^{r(T-t)}$$

$$dG = (\mu - r)G dt + \sigma G dz$$

2.  $G = \ln S$

$$dG = \left( \mu - \frac{\sigma^2}{2} \right) dt + \sigma dz$$